Iterative Predictors of Water Rocket Flight Events

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Background and Justification

Compared to other rockets, either with solid or liquid propellants, water (soda or PET[1] bottle-) rockets have some very specific characteristics:

Their propellant is highly compressed air, thus expelling water and air for thrust.

The typical weight decrease during thrust phase is mainly caused by water expulsion.

The duration of water expulsion phase (thrust phase I) is extremely short - 20 to 50 milliseconds - causing a tremendous acceleration of up to 150 times earth acceleration.

Acceleration attains its maximum at the event of 'Water Out'. At this moment the velocity curve passes through its point of inflection changing from a concave up to a concave down shape.

Thereafter follows a little longer phase (thrust phase II) of exhaust of the remaining pressurized excess air. During this phase the rocket speeds up to its maximum velocity while thrust corrected for gravity and air drag as well as rocket acceleration fall back to zero.

The coast phase (phase III) of a water rocket has a duration of up to 20 times thrust phase. During this phase the rocket flies up to its maximum altitude, the apogee. On its way it slows down due to gravity and air drag.

The descent (phase IV) lasts the longest time in a water rocket flight. It is characterized by the interaction between earthwards directed gravity and skywards directed air breaking. Once both forces become equal the rocket falls at constant descent velocity.

Even after return to earth, at 'Touch Down', there remains a fog of evenly dispersed water droplets inside the PET bottle.

It is the combination of compressed air and water that makes flight predictions of thrust phases I and II extremely difficult, suggesting the necessity of a step-by-step iterative approach. The coast phase (phase III) and descent (phase IV) are directly comparable to those of other rockets. They can be estimated either way: by iteration or calculation.

A surprising number of authors has taken up the challenge to produce user-friendly simulator programs for water rockets[2]. Unluckily, most of the algorithms actually used remain a hidden secret. Revealing the source codes of spreadsheets would give access to functional thinking, learning and comparing. It is the purpose of this paper not to hide but to fully describe an iterative spreadsheet covering the entire flight of a water rocket.

Fig. 1: Shootinger water rocket

Iterative Spreadsheet Approach

As a practical example we take the flight of a Shootinger Water Rocket[3] using an Excel worksheet.

Basic Data

Tab.1 shows the time step as an arbitrary constant and the rest as constants as they can be retrieved from textbooks.

Constants				
Cell	**Parameter**	**Name**	**Value**	**Unit**
E13	Timer	Δt	0.0001	s
E14	Earth Acceleration	g	9.80665	m/s^2
E15	Standard Atmosphere	ATM	1.013	bar
E16	Air Density	ρ_A	1.223	kg/m^3
E17	Water Density	ρ_W	1000	kg/m^3
E18	Adiabatic Constant	κ	1.4	

Tab.1: Constants

The iteration frequency of 10 kHz ($\Delta t = 0.0001\, s$) is extremely high but, due to the very short thrust phases I and II, worthwhile, as we will see in the detailed thrust and acceleration curves.

Tab.2 shows the specific inputs for the Shootinger water rocket.

Variables				
Cell	**Parameter**	**Name**	**Value**	**Unit**
E4	Excess Pressure	P	6	bar
E5	Rocket Volume	VOL	1.5	$liter$
E6	Nozzle Diameter	ND	0.022	m
E7	Rocket Diameter	RD	0.09	m
E8	Drag Coefficient	c_d	0.7	
E9	Rocket Mass	m_C	0.214	kg
E10	Water Mass	m_W	0.4	kg

Tab.2: Variable inputs for a Shootinger water rocket

Using the data of Tab.1 and 2 we get in Tab.3 the parameters as they are used in the iteration.

Calculated Parameters					
Cell	**Parameter**	**Name**	**Formula**	**Value**	**Unit**
E21	Start Pressure	PA	$P \cdot 100000$	600 000	Pa
E22	Nozzle Cross Section Area	NA	$(ND/2)^2 \cdot \pi$	0.00038	m^2
E23	Rocket Cross Section Area	RA	$(RD/2)^2 \cdot \pi$	0.00636	m^2
E24	Air Drag Constant	k	$½ \cdot \rho_A \cdot c_d \cdot RA$	0.00272	kg/m
E25	Initial Air Volume	VOL_0	$(VOL - m_W)/\rho_W$	0.00110	m^3
E26	Initial Excess Air Mass	IAM	$\rho_A \cdot VOL_0 \cdot P/ATM$	0.00797	kg
E27	Rocket Start Mass	m_0	$m_C + m_W$ + IAM	0.62197	kg

Tab.3: Derived rocket parameters for use in the iteration

Assumptions

Before taking off to the calculations we have to make some basic assumptions:

The mass flows per second μ of water and air during expulsion are not constant but decreasing.

If we accept that the exhaust velocity v_{EX} of water depends on the excess air pressure by equation (1)

$$(1) \qquad v_{EX} = \sqrt{\frac{2 \cdot PA \cdot NA}{\rho_W}};$$

if we further accept that the mass flow μ of water per second depends linearly on the exhaust velocity by

$$(2) \qquad \mu = v_{EX} \cdot \rho_W \cdot NA,$$

then we have to accept that PA, v_{EX} and μ are declining non-linearly throughout the water and air expulsion processes. This stays in sharp contrast to other rockets where exhaust velocity and mass flow can be assumed to remain constant or variable by guiding devices.

Indeed, as shown in eqn. (3), there is a strong interaction between exhaust velocity and pressure decrease through the increase of air volume up to the entire volume of the PET bottle:

$$(3) \qquad PA_1 = PA_0 \cdot \left(\frac{AIRVOL_0}{AIRVOL_1}\right)^{\kappa}$$

$$AIRVOL_1 = AIRVOL_0 + \frac{m_{W0} - m_{W1}}{\rho_W}$$

$$m_{W1} = m_{W0} - \mu_1 \cdot \Delta t$$

$$AIRVOL_1 = AIRVOL_0 + \frac{m_{W0} - (m_{W0} - \mu_1 \cdot \Delta t)}{\rho_W}$$

$$AIRVOL_1 = AIRVOL_0 + \frac{\mu_1 \cdot \Delta t}{\rho_W}$$

$$\mu_1 = NA \cdot \rho_W \cdot v_{EX1} \cdot \Delta t$$

$$AIRVOL_1 = AIRVOL_0 + \frac{NA \cdot \rho_W \cdot v_{EX1} \cdot \Delta t}{\rho_W}$$

$$AIRVOL_1 = AIRVOL_0 + NA \cdot v_{EX1} \cdot \Delta t$$

$$PA_1 = PA_0 \cdot \left(\frac{AIRVOL_0}{AIRVOL_0 + NA \cdot v_{EX1} \cdot \Delta t} \right)^{\kappa} .$$

This feedback mechanism of volume and pressure causes a curved asymptotic decline with a steep slope at onset, flattening by time. This applies to excess air pressure PA, exhaust velocity v_{EX}, mass flow μ and, through the relation

(3) $T = 2 \cdot PA \cdot NA$,

to the thrust T of the rocket, knowing that thrust is the core parameter of rocketry. This force has to be corrected for the forces of gravity $m_R \cdot g$ and air drag $k \cdot v^2$:

(4) $T_C = T - m_R \cdot g - k \cdot |v| \cdot v$

The strange formulation of $|v| \cdot v$ instead of v^2 is necessary to assign a negative sign to the square of v once velocity, due to descent, gets a negative sign[4]. By this the true effect of upwards directed air breaking is secured mathematically.
Following Newton's laws, T_C as a force is the basis of calculation of the rocket's acceleration, velocity and altitude taking into account the diminishing weight of the rocket m_R during the thrust phases:

(5) Thrust T_C: $T_C = m_R \cdot a$;

(6) Acceleration a: $a = \frac{T_C}{m_R}$;

(7) Velocity v: $\Delta v = a \cdot \Delta t$;

(8) Altitude h: $\Delta h = \Delta v \cdot \Delta t$.

Whatever air pressure we measure: Under no-vacuum conditions it is always the difference between the pressure examined and the ambient atmospheric air pressure *ATM*.
On the other hand, standard atmosphere *ATM* plays an important role in estimating the additional weight of excess air pumped into the PET bottle: If we put on a balance the 1.5 liter PET bottle containing 0.4 liter water, we weigh the mass of the bottle plus the water mass but not the

$$\rho_A \cdot 1.1/1000 = 1.3$$

grams of air mass in the remaining (1.5-0.4) = 1.1 liter volume VOL_0, because there is no difference between outside and inside air pressure.
If we then pump slowly - to avoid a temperature increase inside the bottle - additional air into the bottle up to the pressure of, say, P = 6 bar we have, in addition, the multiple of P/ATM air mass of 1.3 grams in our bottle. Therefore, our initial excess air mass *IAM* amounts to

$$\rho_A \cdot VOL_0 \cdot \frac{P}{ATM} = 1.223 \cdot 0.0011 \cdot \frac{6}{1.223} \approx 0.008\ kg$$

and makes our bottle this much heavier[5]. Inversely, if released, these 8 grams of excess air mass expand to

$$0.008/1.223 \cdot 1000 \approx 6.5$$

liters of *ATM* air volume.

Therefore, we should not neglect the non-negligible little weight of additional excess air mass in our calculations.

Otherwise, the flow of air mass cannot be considered mathematically.

Although the program assumes a clear-cut transition between thrust phase I and II, there is low probability of excess air 'waiting' until the last drop of water leaves the bottle. High-speed pictures may reveal: After take off (release) water becomes opaque and forms a downward directed cone.

Therefore, we should reckon with a more and more diluted water-air-mixture expelling vapor in the end.

Even after 'Touch Down' there remains a cold fog of condensed air humidity inside the bottle.

Iteration

Tab. 4 shows the denominations and values of the first iteration row serving as a reference for the second row.

Iteration Parameters: First Row

Cell	Parameter	Name	Value
A45	Step	n_0	*0*
B45	Time	t_0	*0*
C45	Exhaust Velocity	v_{EX0}	*0*
D45	Pressure	PA_0	*600 000*
E45	Mass Flow	μ_0	*0*
F45	Water	m_{W0}	*0.4*
G45	Excess Airkg	IAM	*0.00797*
H45	Excess AirVol	VOL_0	*0.0011*
I45	Rocket Mass	m_0	*0.62197*
J45	Thrust	T_0	*0*
K45	Thrust corrected	TC_0	*0*
L45	Acceleration	ACC_0	*0*
M45	Velocity	v_0	*0*
N45	Altitude	ALT_0	*0*

Table 4: Parameters of the first iteration row

The second row, shown vertically in Tab. 5, is the core piece of the iteration program because it is copied down unchanged 1740 times to cope with the events of 'Water Out', 'Maximum Velocity', 'Excess Air Out', 'Apogee' and 'Touch Down'.
To save space after 'Excess Air Out', the timer changes from 0.0001 to 0.05 seconds. The *if*-conditions make sure that air density takes over as soon as water mass is zero. The *max(...,0)*-conditions protect some parameters against nonsense negative values.

Iteration Parameters: Second Row

Cell	Parameter	Name	Formula
A46	Step	n_1	n_0+1
B46	Time	t_1	$if(AIR_0>0, t_0+\Delta t, t_0+0.05)$
C46	Exhaust Velocity	v_{EX1}	$if\left(WATER_0>0, \sqrt{2\cdot PA_0/\rho_W}, \sqrt{2\cdot PA_0/\rho_A}\right)$
D46	Pressure	PA_1	$if(WATER_0>0\#or\#AIRKG_0>0, PA_0\cdot(AIRVOL_0/(AIRVOL_0+NA\cdot v_{EX1}\cdot\Delta t))^{\kappa}, 0)$
E46	Mass Flow	μ_1	$if(WATER_0>0, v_{EX1}\cdot NA\cdot\rho_W, v_{EX1}\cdot NA\cdot\rho_A)$
F46	Water	$WATER_1$	$max(WATER_0-\mu_1\cdot\Delta t, 0)$
G46	Excess Air Mass	$AIRKG_1$	$max(if(PA_1=0, 0, AIRKG_0-v_{EX1}\cdot NA\cdot\rho_A\cdot\Delta t), 0)$
H46	Excess Air Volume	$AIRVOL_1$	$AIRVOL_0+(WATER_0-WATER_1)/\rho_W$
I46	Rocket Mass	MR_1	$MC+WATER_1+AIRKG_1$
J46	Thrust	$THRUST_1$	$2\cdot PA_1\cdot NA$
K46	Thrust corrected	$THRUSTC_1$	$THRUST_1-MR_1\cdot g-k\cdot\lvert v_0\rvert\cdot v_0$
L46	Acceleration	ACC_1	TC_1/MR_1
M46	Velocity	v_1	$if\begin{pmatrix}AIRKG>0, v_0+ACC_1\cdot\Delta t,\\ if(ALT_0=0, v_0+ACC_1\cdot 0.05)\end{pmatrix}$
N46	Altitude	$ALTITUDE_1$	$max\left(if\begin{pmatrix}AIRKG_0>0, ALT_0+v_1\cdot\Delta t,\\ ALT_0+v_1\cdot 0.05\end{pmatrix}, 0\right)$

Table 5: Second row of iteration.

Iteration Results

Table 6 shows the first 3 rows of the actual iteration.

n	t	v_{ex}	PA	μ	Water	Excess Airkg	Excess AirVol	m_R	Thrust	Thrust corrected	Accel	Velocity	Altitude
0	0	0	600000	0	0.400	0.0079682	0.0011000	0.622	0	0	0	0	0
1	0.0001	34.64	598996	13.168	0.399	0.0079666	0.0011013	0.621	456	450	725.164	0.073	0.000007
2	0.0002	34.61	597995	13.157	0.397	0.0079650	0.0011026	0.619	455	449	725.494	0.145	0.000022
.	.	.	.	.	.	.		.	.	.			.
Tendency		↓	↓	↓	↓	↓	↑	↓	↓	↓	↑	↑	↑

Table 6: First three rows of water rocket iteration

The first row shows the same initial values as Tab. 3. Time step 1 is assumed to be the rocket launch. The comparison of the values of the second row with those of the third row reveals the trends as shown by arrows on the bottom of the table. As the arrow in the column 'Excess Airkg' indicates, a little allowance is given to the air mass to loose its weight even during water expulsion at the rate of water exhaust velocity.

Fig. 2 shows the iterated thrust curves of T and T_C during thrust phases I and II. The nearly identical graphs of these two parameters signify that the correction for gravity and air drag at this stage only has a marginal effect.

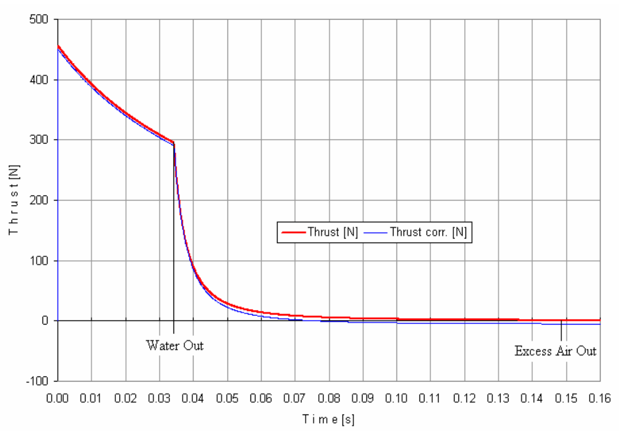

Fig. 2: Thrust curve during phases I and II

The graphs shows how, within 34 milliseconds, thrust falls almost linearly down to 65 per cent of its initial value due to water expulsion. Then, when the 1000 times more volatile air exhaust takes over, thrust reduces to a mere 0.4 per cent at the end of the following 36 milliseconds. From then on, with virtually no propellant left, gravity and air drag prevail and let T_C fall below zero. The tiny rest of excess air oozes out without effect.

Fig. 3 shows the resulting acceleration, velocity and altitude curves during thrust phases I and II.

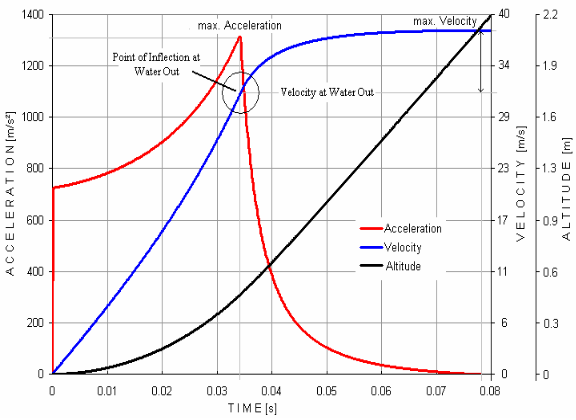

Fig 3: Acceleration, velocity and altitude during thrust phases I and II

The reason for the point of inflection of the velocity curve is not that thrust has gone down to zero, as it would be the case for the second derivative in textbook calculus, but that there is simply no substantial water mass left in the bottle and that the still highly pressurized excess air is taking over. This causes acceleration not to fall back to zero at once but to decline smoothly by an elegant curve.

During thrust phase II velocity gains another 20 per cent on top of its 'Water Out' value.

After an exponential increase in phase I, altitude raises linearly during phase II (Fig. 3). Thereafter, during coast phase, the slope of the altitude curve dwindles down to zero at apogee (Fig. 4).

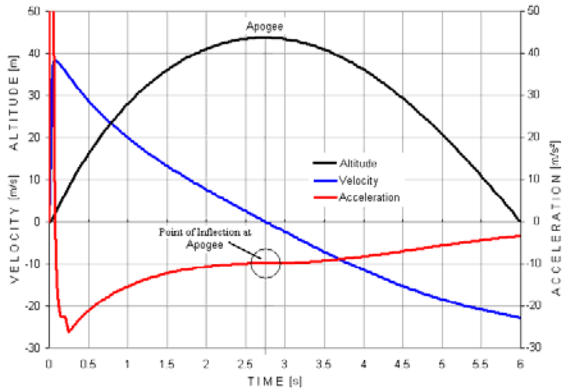

Fig. 4: Acceleration, velocity and altitude during coast and descent phases

Due to the rocket's high velocity, air drag displays its strongest negative effect at the beginning (Fig. 5). This forces acceleration down to its minimum (Fig. 4). Then, due to diminishing velocity, it becomes less negative and reaches, at the moment of maximum altitude, a flat point of inflection where the rocket's acceleration is equal to negative earth acceleration. At this moment, there is nothing left but $-g$ in eqn. (6). Velocity has slowed down to zero. Accordingly, as indicated in Fig. 5, air drag is zero too. Simultaneously, thrust T_C is equal to the rocket's gravity.

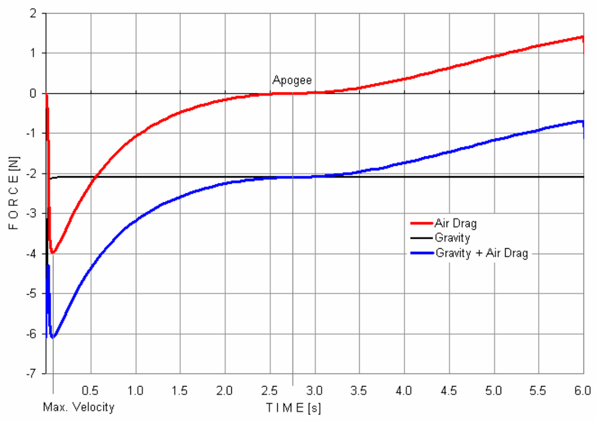

Figure 5: Air drag, gravity and thrust during coast and descent phases

Thereafter, while the water rocket falls back to earth, velocity assumes a negative sign, and air drag is breaking it. The altitude of the apogee is not sufficient - it would have needed 300 m at least - to fall at a constant descent velocity.
Tab. 7 summarizes the flight events of the water rocket. This table shows clearly: The events of 'Water Out' and 'Excess Air Out' do not coincide with the water rocket's maximum velocity.

Event	t	v_{ex}	PA	μ	Water	Excess Airkg	Excess AirVol	m_R	Thrust	Thrust corrected	Accel	Velocity	Altitude
Water Out	0.0342	28	388299	10.60	0	0.0075	0.0015	0.221	296	291	1313	31	0.5
Max Velocity	0.0764	115	8017	0.05	0	0.0024	0.0015	0.216	6	0	0	38	2.0
Excess Air Out	0.1488	46	1314	0.02	0	0	0.0015	0.214	1	-5	-23	37	4.8
Max Altitude	2.7488	0	0	0	0	0	0.0015	0.214	0	-2	-10	0	43.8
Touch Down	5.9988	0	0	0	0	0	0.0015	0.214	0	-1	-3	-23	0

Table 7: Events of a water rocket flight resulting from 1605 iterations

This is in agreement with results from Dean Wheeler's simulator. Therefore, we cannot reckon with classical 'burnout velocities' in water rockets.
On the other hand, conventional calculations, say Fehskens-Malewicki equations, essentially need the exact burnout time and burnout velocity to predict apogee altitude and time.
Tab. 8 gives a comparison between iterated and calculated predictions of water rocket flight events. From this table we conclude that especially estimates of thrust phase II events are highly divergent making iterations inevitable. All other events can be estimated either way without grossly loosing information.

Chacteristic	Iterated	Calculated	Unit	Diff %
Water Out Time	0.034	0.030	s	11.2
Water Out Altitude	0.481	0.446	m	7.3
Water Out Velocity	31.122	35.395	m/s	-13.7
Water Out Air Pressure	388299	388663	N/m²	-0.1
Excess Air Out Time	0.149	0.052	s	65.1
Excess Air Out Altitude	4.776	0.554	m	88.4
Excess Air Out Velocity	37.020	45.463	m/s	-22.8
Maximum Altitude	43.755	45.274	m	-3.5
Time to Apogee	2.749	2.795	s	-1.7
Touch Down Time	5.999	6.132	s	-2.2
Touch Down Velocity	-22.748	-22.961	m/s	-0.9

Tab. 8: Comparison between iterated and calculated water rocket flight events.

[1] Polyethylen-Terephthalat PET with polyester structure:
http://www.psrc.usm.edu/macrog/pet.htm
[2] e.g. Dean R. Wheeler 2002:
http://www.et.byu.edu/~wheeler/benchtop/
Clifford Heath 2001:
http://polyplex.org/cjh/rockets
Bruce Berggren 2002:
http://www.geocities.com/wrgarage/
[3] http://academy-europe.de/divhtm/18101.htm
[4]Peter Nielsen 1999:
http://www.ent.ohiou.edu/~et181/rocket/Nielsen_Rocket.pdf
[5]PISA-Der Ländertest September 10,2005: Gewicht der Luft ; www.wdr.de/tv/pisa