A SHORT STORY ABOUT THE HODOGRAPH

by

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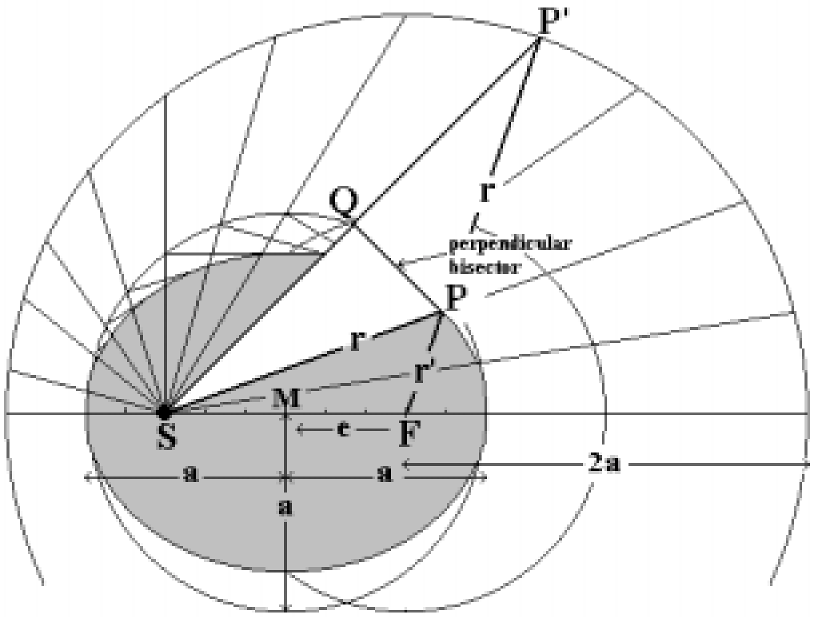
An extraordinary event: In a lecture on 13th of March 1964 the famous professor Richard P. Feynman explains to freshmen why the planets do not orbit on circles around the Sun but on ellipses (Goodstein and Goodstein, 1996). He declares his proof as ‘elementary’ meaning that he does not use already known methods, especially not infinitesimal calculus.

After explaining how to construct an ellipse by the ‘gardener’s method’ using a thread whose ends are fixed on two poles, the foci, Feynman shows that the sum of the two radius vectors *r + r’* of the ellipse is constant. From physics he quotes that light emitted from one focus *S* of an elliptic mirror wall is reflected into the alternate focus *F*. Feynman uses this property to prove that the normal to any tangent of an ellipse halves the angle between *r* and *r’*. Feynman then derives Kepler’s second law of ‘equal areas are swept out at equal times’ according to Newton’s Proposition No.1 book1 of the ‘Pricipia’, and the inverse square law, to arrive at the main point of his proof, a leading circle for the elliptic orbit of a planet which turns out to be the principle of the hodograph (Fig.1).

Figure 1
Construction of an Ellipse based on a Circle and an Eccentric Point *S*



He attributes his knowledge to Mr Fano who, according to Derbes (2001), had reinvented the ‘hodograph’. The idea was first published by Sir Wiliam Rowan Hamilton in 1874 who created the name ‘hodograph’ as a ‘descriptor of the way’, derived from the Greek words *ο(δο&φ* *a way* and *γρα&φειν* *to describe*. What is this hodograph? The following lines will show that it is a combination of velocity vectors of a planet and its elliptic orbit which can be used to determine its physical properties.

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Two concentric circles with radii *a* and *2a* and the common centre *F* are given. If we move the smaller circle to the left by the distance *e* , we move its centre from *F* to *M*. By this we can construct an ellipse as an affine projection of the smaller circle $(r = a)$. The principal focus *S* is located at the distance *2e* from *F*. *F* becomes the alternate focus of the ellipse. At the same time, *F* remains the centre of the leading circle. Radius vectors are emitted from *S*, intersecting with the leading circle $(r = 2a)$ at any point *P'*. The perpendicular bisector to *SP'* cutting *SP'* at *Q* is the tangent of an ellipse. Fig.1 shows that the circle around $M\ (r = a)$ is the locus of these points *Q* . The points *P* of the tangents are points on the ellipse. The figure shows that the contour of the ellipse appears by a large number of such tangents. Thus, the major axis of the ellipse *2a* is equal to the radius ρ of the leading circle.

To learn about its properties we take a simple model of an orbital ellipse. The principal focus is located at the Sun *S (0|0)* according to Kepler's first law. The shortest distance (perihelion) between the Sun *S* and the planet *P* is set to $r_1 = 0.5\ LU$ (length unities), the longest distance (aphelion) to $r_2 = 1.0\ LU$. The following table shows the characteristics of this model ellipse.

Table 1
Formulas and Numerical Values of the Characteristics of a Model Planetary Orbit
$(r_1 = 0.5LU;\ r_2 = 1.0LU)$

Parameter	Formula	Value	Parameter	Formula	Value
semi axis major	$a = \frac{r_1 + r_2}{2}$	$\frac{3}{4}$	linear eccentricity	$e = \frac{r_2 - r_1}{2}$	$\frac{1}{4}$
semi axis minor	$b = \sqrt{r_1 \cdot r_2}$	$\sqrt{\frac{1}{2}}$	numerical eccentricity	$\varepsilon = \frac{r_2 - r_1}{r_1 + r_2}$	$\frac{1}{3}$
semi latus rectum	$p = 2 \cdot \frac{r_1 \cdot r_2}{r_1 + r_2}$	$\frac{2}{3}$			

This shows that the lengths of the perihelion and the aphelion are sufficient to describe the geometrical properties of an ellipse.

From Kepler's second law we know that the planet attains its highest speed v_{max} at the perihelion r_1 and the minimum v_{min} at the aphelion r_2 . The area constant *B* can be defined by the product of the maximum velocity times the minimum distance which is equal to the minimum velocity times the maximum distance (1).

(1) $$v_{max} \cdot r_1 = v_{min} \cdot r_2 = B$$

The tangents to the ellipse are perpendicular to the main axis at the two main apexes. Inversely, we can determine the minimum and maximum velocities of the planet if we know the magnitude of the area constant (2).

(2) $$v_{max} = \frac{B}{r_1}\ ;\ v_{min} = \frac{B}{r_2}\ ,$$

where *v* indicates the length of a velocity vector.

In Fig.1 we see that the diameter of the leading circle or hodograph is set to twice the main axis of the orbital ellipse. Therefore, in terms of velocity vectors, the diameter 2ρ of this circle is equal to the maximum velocity plus the minimum velocity (3).

$$(3) \qquad 2\rho = \frac{B}{r_1} + \frac{B}{r_2} = \frac{B \cdot (r_1 + r_2)}{r_1 \cdot r_2} = \frac{B \cdot 2a}{b^2} = \frac{2B}{p}$$

Since, on the other hand, the radius ρ of the hodograph is set equal to the main axis $2a$ of the orbital ellipse, the area constant B is given by (4).

$$(4) \qquad B = 2a \cdot p = 2b^2$$

Equation (4) assures the link between the hodograph and the orbit. Feynman does not mention this equation. To the question of a student asking him about the physical significance of the radius ρ, as we hear it on the lecture CD, we could answer that the radius ρ of the hodograph is set equal to the arithmetic mean of the velocities of a planet. With the geometric mean of the extreme velocities $v_c = \sqrt{v_{max} \cdot v_{min}}$, the planet would orbit on a circle $(r = a)$.

In addition, GM as the product of the gravitational constant G times the mass M of the 'Sun' can now be derived from the area constant B and the semi latus rectum p by equation (5).

$$(5) \qquad GM = \frac{B^2}{p}$$

Table 2 sums up the numerical results which can be obtained from the hodograph..

Table 2
Formulas and Numerical Values of the Physical Properties of a Model Planetary Orbit
$(r_1 = 0.5\,LU;\ r_2 = 1.0\,LU)$

Parameter	Formula	Value	Parameter	Formula	Value
area constant	$B = 2a \cdot p$	1	maximum velocity	$v_{max} = B/r_1$	2
gravit.const.˙ mass	$GM = \frac{B^2}{p}$	$\frac{3}{2}$	minimum velocity	$v_{min} = B/r_2$	1
energy	$A = -\frac{GM}{2a}$	-1	time for 1 revolution	$T = 2\pi \cdot \sqrt{\frac{a^3}{GM}}$	$\pi\ \frac{3}{2}\sqrt{\frac{1}{2}}$

By equation (6) we can determine the length of any velocity vector at any position along the orbital ellipse.

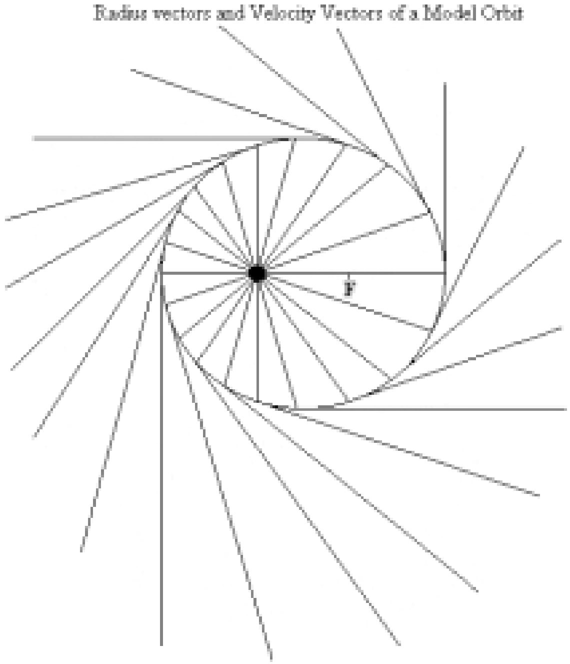
$$(6) \qquad v = \sqrt{GM \cdot \left(\frac{2}{r} - \frac{1}{a}\right)}$$

Since the velocity vectors are tangents to the orbital ellipse we have to know their slopes. These can be calculated by equation (7).

$$(7) \qquad m_t = \frac{b^2}{a^2} \cdot \frac{x_p + e}{y_p}$$

The Fig.2 shows the velocity vectors indicating that the planet slows down with increasing distance from the Sun S and how it accelerates as soon as the distance gets shorter. It also shows that the angle between neighbouring vectors remains constant.

Figure 2
Radius vectors and Velocity Vectors of a Model Orbit



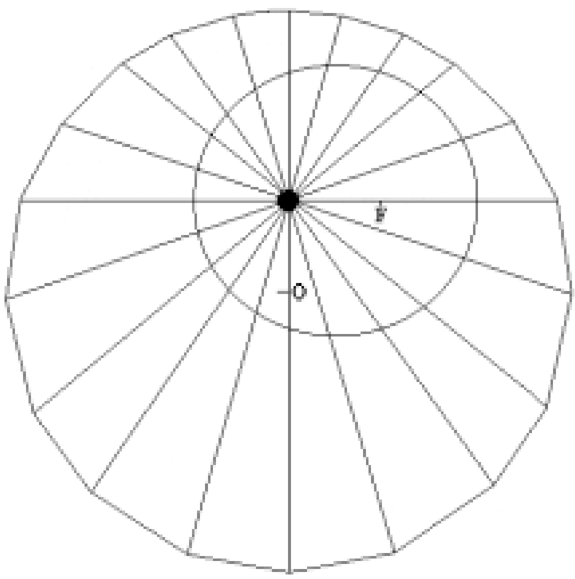
The graph shows that the maximum and minimum velocities are perpendicular to the major axis, pointing to opposite directions.

By parallel translation we unite the origins of the velocity vectors in S without changing their lengths. The hodograph is then created by connecting the end points of the vectors. It turns out, although somewhat blurred by an optical effect, that it becomes a perfect circle, as Hamilton has postulated for any conic section trajectory (Fig.3).

The maximum velocity v_{max} at the perihelion and the minimum velocity v_{min} at the aphelion are now on one line. In Cartesian coordinates we find the centre O of the hodograph at $x = 0$; the y-value being the difference between v_{max} and $\boldsymbol{r}$, as expressed in (8).

$$(8) \qquad y_O = \left(v_{max} - \rho\right) = \frac{B}{p} \cdot \varepsilon = 2e\,.$$

Figure 3
Concentrating the Velocity Vectors at the Main Focus

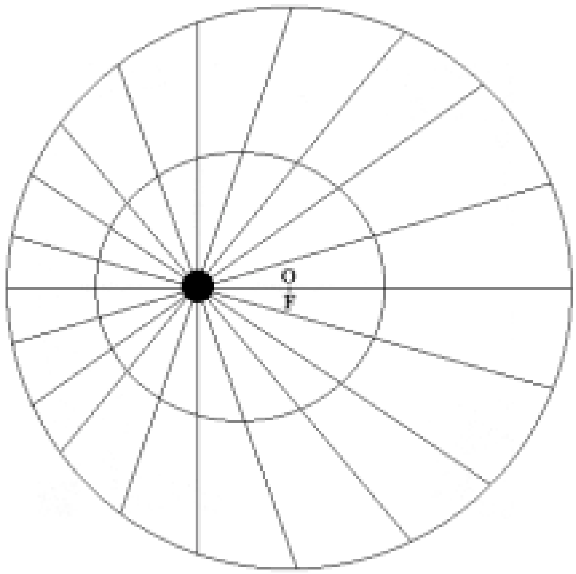


This is, at the same time, the distance $2e$ between the two foci.

On the other hand, Fig.3 shows that the inverse procedure, designing an elliptic orbit from the hodograph is not yet possible. We, therefore, need a trick as James Clerk Maxwell used it in his book ‘Matter and Motion’: We turn the hodograph by $90°$ around the focus S, with the effect that v_{max} now appears as an extension of r_2 and v_{min} as an extension of r_1, as shows Fig.4.

Finally the radius of the hodograph is equal to the major axis of the ellipse ($\rho = B/p = 2a$) due to the same distance between *S* and *O* and *S* and *F*.

Figure 4
Turning the Hodograph by 90° around the Main Focus



In a footnote Hamilton supposes that the aberration which is caused by the speed of the Earth on its orbit around the Sun, describes a hodograph in the run of one year. This contradicts the common interpretation, saying that the aberration is a projection of the Keplerian elliptic orbit of the Earth. Who is right?

Literature

Derbes, D. 2001: Reinventing the wheel: Hodographic solutions to the Kepler problems. Am.J.Phys. 69 (4), 2001.

Goodstein, D.L. and J.R. Goodstein 1996: Feynman’s Lost Lecture. The Motion of Planets Around the Sun. New York; Lodon. W.W. Nrton & Co. (1996).

Guthman, A. 1994: Einführung in die Himmelsmechanik und Ephemeridenberechnung. Mannheim; Leipzig; Wien; Zürich. BI-Wiss.-Verl.

Hamilton, W.R. 1847: The Hodograph, or a new method of expressing in symbolical language the Newtonian Law of Attraction. Edited by David R. Wilkins. Proceedings of the Royal Irish Academy, 3 (1847), pp.\344-353.

Hilbert, D. and S. Cohn-Vossen 1932: Anschauliche Geometrie. 2. Aufl. Berlin; Heidelberg. Springer (1996).

Maxwell, J.C. 1920: Matter and Motion. Mineola; Dover Pulications (1999).

Newton, I. 1726: The Principia: mathematical principles of natural philosophy. A new translation by I.B. Cohen and A.Whitman. Berkeley; Los Angeles. Univ. of California Press (1999).